

Experimental simulation of fractional statistics of abelian anyons in the Kitaev lattice-spin model

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In two-dimensions, the laws of physics even permit the existence of anyons which exhibit fractional statistics ranging continuously from bosonic to fermionic behaviour. They have been responsible for the fractional quantum Hall effect and proposed as candidates for naturally fault-tolerant quantum computation. Despite these remarkable properties, the fractional statistics of anyons has never been observed in nature directly. Here we report the demonstration of fractional statistics of anyons by simulation of the first Kitaev lattice-spin model on a nuclear magnetic resonance system. We encode four-body interactions of the lattice-spin model into two-body interactions of an Ising spin chain in molecules. It can thus efficiently prepare and operate the ground state and excitations of the model Hamiltonian. This quantum system with convenience of manipulation and detection of abelian anyons reveals anyonic statistical properties distinctly. Our experiment with interacted Hamiltonian could also prove useful in the long run to the control and application of anyons.

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Particles in nature behave at two distinct statistics referring to bosons and fermions. But if one is restricted to two-dimensional (2D) systems, a class of fractional statistics intervening between bosonic and fermionic statistics appears. It is determined by some quasi-particles known as anyons [1, 2]. The quantum state of anyons can acquire an unusual phase when one anyon is exchanged with another one, in contrast to usual values +1 for bosons and -1 for fermions. In general, the quantum state of the n indistinguishable particles belongs to a Hilbert space that transforms as a unitary representation of the braid group. Abelian anyons are defined to be these particles that transform as one-dimensional representation of the braid group, while the nonabelian anyons correspond to representations that are of dimension greater than one [3].

Anyons have intrigued great interests not only for their unusual fractional statistics but rather for reason of their crucial role in topological quantum computation (TQC) [4, 5, 6, 7, 8, 9, 10], since exotic exchange statistics on 2D plane endows anyons with nontrivial topological properties that indicates their potential values in use. Although anyons have been responsible for the fractional quantum Hall system [11, 12], a direct observation of fractional statistics associated with anyon braiding is hard in this system and has attracted several intriguing theoretical proposals [13, 14, 15, 16] etc.. Particularly, the exactly solved models for naturally fault-tolerant topological quantum computation proposed by Kitaev [4, 5] demonstrate the excitations with anyonic features and thus also provide a direction for experimental detection of anyons [17].

Recently, trapped ions, photons, or nuclear magnetic resonance (NMR) were suggested to realize a small-scale system for proof-of-principle demonstration of the anyon

braiding statistics based on the first Kitaev lattice-spin model [18]. Experimental demonstrations by using photons have been made with four [19] or six [20] photons. Yet since the background Hamiltonian vanishes in such optical systems, the defect that they are not protected from noise and the particle interpretation of the excitations is ambiguous was soon pointed out [21].

In this letter, inspired by the four-photon experiment [19], we start by simulation of the first Kitaev lattice-spin model with the interacted nuclear spin-1/2 particles in molecules to demonstrate the fractional statistics of anyons. A significant advantage of using nuclear spin system is that the concept of anyons is originally and generally considered with the interacted spin particles in magnetic field and there are well established nuclear magnetic techniques for coherent control and measurement of nuclear spin systems [22]. This allows us to prepare the initial multi-spin entangled state of spins, directly simulate dynamic evolution of anyonic quasi-particles with the Kitaev lattice-spin model Hamiltonian, and detect the abelian anyons.

First let us review the essence of the first Kitaev lattice-spin model [4]. Considering a $k \times k$ square lattice on a torus (see Fig. 1 (a)), one spin or qubit is attached to each edge of the lattice. Thus there are $2k^2$ qubits. For each vertex s and each face p , consider operators of the following form:

$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x, \quad B_p = \prod_{j \in \text{boundary}(p)} \sigma_j^z,$$

where the σ_j^α denotes the Pauli matrix ($\alpha = x, y, z$) and it acts on the j -th qubit of a star s or boundary p (see Fig. 1 (b)). These operators commute with each other because star s and boundary p have either 0 or 2 common

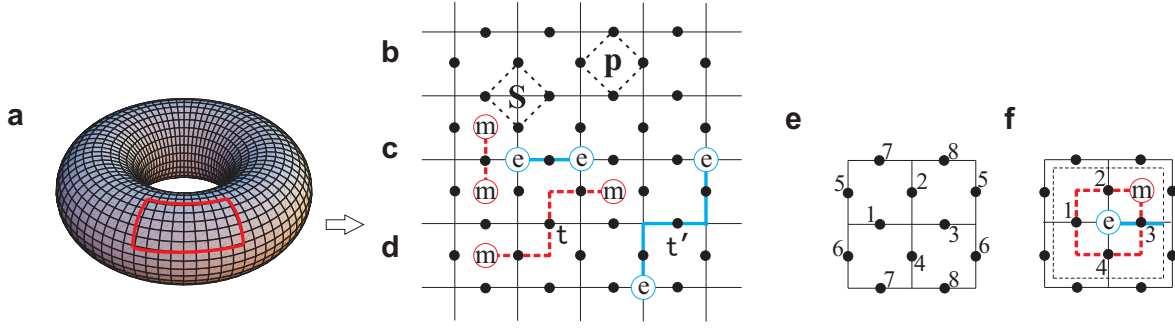


FIG. 1: (Color online) An illustration of the first Kitaev model: (a) A $k \times k$ square lattice residues on a torus; (b) The hermitian operator A_s (B_p) is defined on a star s (face p) and acts on the four qubits of the star (face); (c) Two m particles are created as $|m\rangle = \sigma_r^x |\xi\rangle$ at the both faces around the r -th qubit and two e particles are created as $|e\rangle = \sigma_r^z |\xi\rangle$ at the both vertices around the r -th qubit; (d) The string operator $S(t)$ acting on the ground state creates two e (m) particles around the first acted qubit and then move one particle along the path t (t'); (e) The minimal system is a 2×2 square lattice with eight qubits on edges; (f) A braiding process can be achieved with only four qubits.

qubits. The operators A_s and B_p represent four-body interactions and they are Hermitian with eigenvalues 1 and -1 . Putting them together, it constructs the model Hamiltonian

$$H = - \sum_{s \in \text{torus}} A_s - \sum_{p \in \text{torus}} B_p.$$

Its four-fold degenerate ground states $\{|\xi\rangle\}$ satisfy $A_s|\xi\rangle = |\xi\rangle$, $B_p|\xi\rangle = |\xi\rangle$ for all s, p and construct a protected subspace. Hence it can define a quantum code called a *toric code*. Operators A_s and B_p are thus called the *stabilizer operators* of this code.

Here we concern more about anyons as the excitations of this Hamiltonian. A pair of ‘electric charges’ living on vertices (or ‘magnetic vortices’ living on faces) will be created at the both sides of the r -th qubit by the operator σ_r^z (or σ_r^x) acting on the ground state $|\xi\rangle$ (see Fig. 1 (c)). The two kinds of particles are denoted by e and m respectively. For the fusion rule ($e \times e = m \times m = 1$), one thus can define the so-called string operators

$$S^z(t) = \prod_{r \in t} \sigma_r^z, \quad S^x(t') = \prod_{r \in t'} \sigma_r^x, \quad (1)$$

where $r \in t$ (t') means the r -th qubit on the path t (t'). The operation $S^z|\xi\rangle$ (or $S^x|\xi\rangle$) first creates two e (or m) particles at the two sides of the primarily acted qubit and then moves one of them to the end of the path t (or t') (see Fig. 1 (d)). If one utilizes the string operators to move a m (or e) particle around an e (or m) particle, we see that the total wavefunction acquires a global phase factor -1 . This is the unusual statistical property of abelian anyons.

From the Fig. 1 (a) and (e), we can see that the minimal system of this Kitaev model is a 2×2 square lattice with eight independent qubits on edges. The braiding statistics can be performed on this plane with a m particle winding one cycle around the other e particle. Such a

series of operations only are related to, for example, the 1-st, 2-nd, 3-rd, 4-th qubits. So we can reduce the system to include merely these necessary qubits and it will not compromise the physical meaning. In our experimental setup, we would use such four-qubit system which could be viewed as a small piece from a large one (Fig. 1 (f)).

The Hamiltonian for this four-qubit system is $H = -\sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x - \sigma_1^z \sigma_2^z - \sigma_2^z \sigma_3^z - \sigma_1^z \sigma_4^z - \sigma_3^z \sigma_4^z$ with its ground state $|\xi\rangle = (1/\sqrt{2})(1 + \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x)|0000\rangle$ that is just GHZ state $|\text{GHZ}\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$. In principle, the creating operation of anyons can be performed on any one of the four qubits. Here we choose the 3-rd qubit for the e particle creation, and the 2-nd, 1-st, 4-th, 3-rd qubits in order for m particle creation and transportation. In order to detect the global phase factor in front of the total wavefunction after braiding, one needs to superpose the initial state and the excitation state as $(|\xi\rangle + i|e\rangle)/\sqrt{2}$. Then after braiding the state becomes $(|\xi\rangle - i|e\rangle)/\sqrt{2}$. It has the effect of transforming the unobservable global phase factor into a visible local phase factor. By measuring the relative phase factor before and after the braiding process, one can assure the fractional statistics of abelian anyons.

The experimental network was shown as Fig. 2. In the experiment, we first prepare a four-qubit GHZ state $|\text{GHZ}\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$; then implement a \sqrt{Z} operation that changes the initial ground state to a superposed state $(|\xi\rangle + i|e\rangle)/\sqrt{2}$; after that, a string operation of Eq. (1) $S^x(t) = \sigma_3^x \sigma_4^x \sigma_1^x \sigma_2^x$ acts on the superposed state to revolve the m particle around e particle one lap and it results the appearance of a local phase factor in the state $(|\xi\rangle - i|e\rangle)/\sqrt{2}$; finally take a \sqrt{Z}^{-1} operation to annihilate the anyons to get the state $(|0000\rangle - |1111\rangle)/\sqrt{2}$. Here the minus sign before the second term arises exactly from the anyons braiding. To measure changes of the state, we use the state tomography technique to display

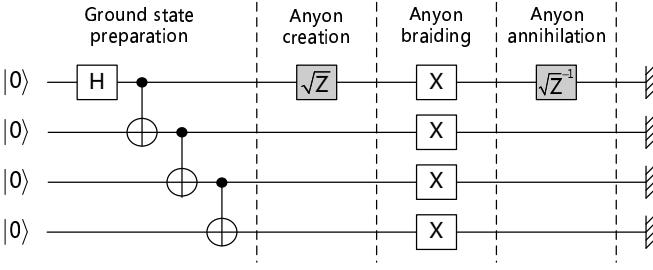


FIG. 2: The quantum network for the experiment. H means a Hadamard operation and Z/X corresponds to σ_z/σ_x operation. For the \sqrt{Z} operation, it has the function of creating anyons and superposing the ground state and the anyonic excitation, while the \sqrt{Z}^{-1} is its inverse operation.

all 16×16 components for a four-qubit density matrix rather than using an ancilla to form interferometry in optics. The relative magnitudes and phase factors between these components can be explicitly demonstrated out. For comparison, we also measure the state change after a revolving of m particle in absence of e particle by canceling the \sqrt{Z} and subsequent \sqrt{Z}^{-1} operations above.

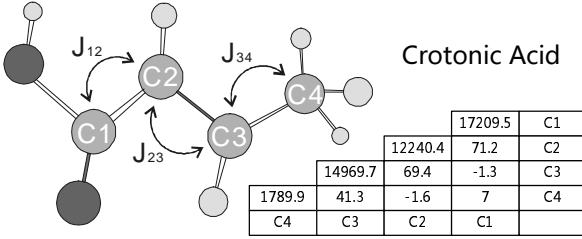


FIG. 3: Molecular structure and Hamiltonian parameters Crotonic acid. The chemical shift of each of the carbon nuclei is given by the corresponding diagonal elements while the coupling strengths are given by the each of the different off-diagonal elements.

In our liquid state NMR system, the molecule (^{13}C -labeled crotonic) used for this experiment contains four ^{13}C spin-1/2 nuclei as qubits, shown as Fig. 3. The Hamiltonian of the 4-qubit system is (in angular frequency units) $H_{sys} = \sum_{i=1}^4 \omega_i I_z^i + 2\pi \sum_{i<j}^4 J_{ij} I_z^i I_z^j$ with the Larmor angular frequencies of the i^{th} spin ω_i and spin-spin coupling constants $J_{12} = 71.2\text{Hz}$, $J_{13} = -1.3\text{Hz}$, $J_{14} = 7\text{Hz}$, $J_{23} = 69.4\text{Hz}$, $J_{24} = -1.6\text{Hz}$, $J_{34} = 41.3\text{Hz}$. The system Hamiltonian H_{sys} exhibits one-dimensional Heisenberg spin chain with neighboring two-body interactions. Experiments were performed at room temperature using a standard 400MHz NMR spectrometer (AV-400 Bruker instrument).

The system was first prepared in a pseudo-pure state (PPS) $\rho_{0000} = \frac{1-\epsilon}{8}\mathbf{I} + \epsilon|0000\rangle\langle 0000|$, where $\epsilon \approx 10^{-5}$ describes the thermal polarization of the system and \mathbf{I}

is a unit matrix, using the method of spatial averaging. The first part of PPS is the background environment, while the second part is the effective pure state in use denoted by ρ_ϵ (not including ϵ). Then to prepare the GHZ state, a Hadamard gate and three CNOT gates were needed, shown as the first part of Fig. 2. Finally, a few single qubit operations were used to do the rotation of m particle with or without e particle, shown as the other part of Fig. 2.

In order to improve the quantum coherent control, experimentally every single qubit gates was created by using robust strongly modulating pulses (SMP) [23, 24, 25]. We maximize the gate fidelity of the simulated propagator to the ideal gate, and we also maximize the effective gate fidelity by averaging over a weighted distribution of radio frequency (RF) field strengths, because the RF-control fields are inhomogeneous over the sample. Theoretically the gate fidelities we calculated for every pulse are greater than 0.99, and the pulse lengths range from 200 to 500 μs . The quantum circuit of Fig. 2 was realized with a sequence of these local SMPs separated by time intervals of free evolution under the Hamiltonian. The overall theoretical fidelity of this pulse sequence is about 0.95. The density matrices of the spin system after the initial GHZ state preparation and after braiding were reconstructed using state tomography, which involves applying 40 readout pulses to obtain coefficients for the 256 operators comprising a complete operator basis of the four spin- $\frac{1}{2}$ ^{13}C nuclei.

The experimental results are displayed on Fig. 4. The ground state $|\xi\rangle$ (i.e., GHZ state) for anyons braiding is prepared at initial time t_0 with fidelity $F(t_0) = \langle \xi | \rho_\epsilon(t_0) | \xi \rangle = 0.93$ and its state tomography is shown on Fig. 4 (a). After braiding, the evolved state $|\xi'\rangle$ at time t_1 obtain a local phase change with a minus sign instead of a positive sign in the initial ground state $|\xi\rangle$ and its state tomography on Fig. 4 has the fidelity $F(t_1) = \langle \xi' | \rho_\epsilon(t_1) | \xi' \rangle = 0.94$. The different directions of bars reveal the phase shift, which is just the requirement of the anyonic fractional statistics. For comparison, we also measure the state $|\xi''\rangle$ after one m particle revolves one lap around empty center at time t_2 and it gives a completely different result from that after braiding, which shows the topological property of anyons. The state tomography is shown on Fig. 4 (c) and has the fidelity $F(t_2) = \langle \xi'' | \rho_\epsilon(t_2) | \xi'' \rangle = 0.98$. Clearly, the results are in excellent agreement with the theoretical expectation. The deviation between the experimental and theoretical values is primarily due to the inhomogeneity of the radio frequency field and the static magnetic field, imperfect calibration of radio frequency pulses, and signal decay during the experiments.

In summary, we are the first to simulate the fractional statistics of the 1/2 abelian anyons by using interacted nuclear spin-1/2 particles in molecules. The experimental results directly demonstrate the presence of fractional

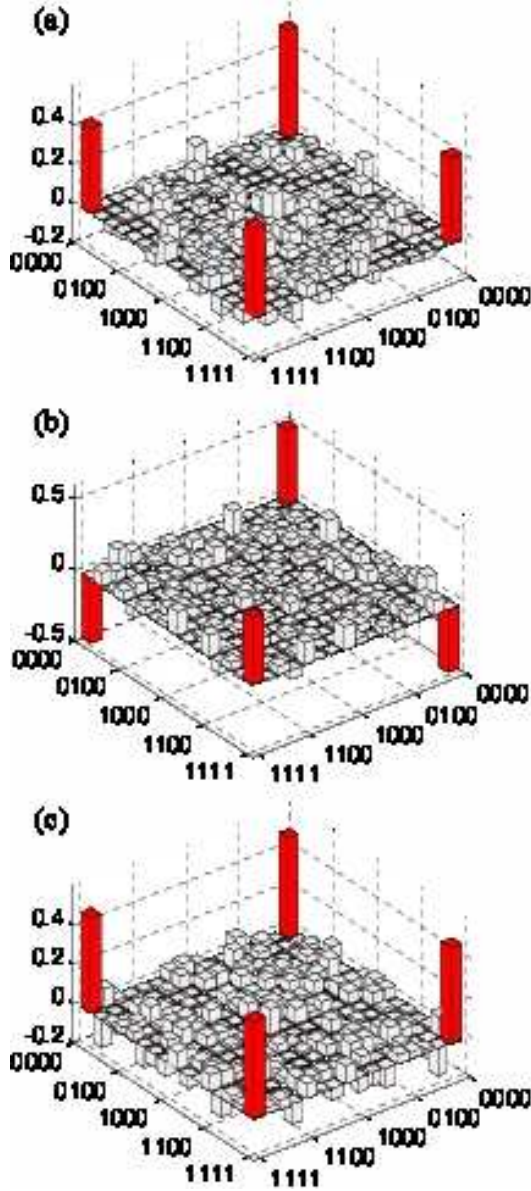


FIG. 4: (Color online) Experimental results are displayed in the form of state tomography: (a) the prepared ground state $|\xi\rangle = (1/\sqrt{2})(|0000\rangle + |1111\rangle)$; (b) the resultant state due to the anyons braiding $(1/\sqrt{2})(|0000\rangle - |1111\rangle)$; (c) the trivial state after a m particle revolving one lap without braiding $(1/\sqrt{2})(|0000\rangle + |1111\rangle)$, which is used for comparison with step (b). Theoretically, the height of the red bars should be $+0.5$ or -0.5 , and other bars should be 0.

statistics in a physical system. With the number increase of controllable qubits to eight, it can realize completely the complete smallest Kitaev model as discussed previously. This kind of proof-of-principle demonstration of anyons in small and relatively simple systems will represent an important step toward the long pursued goal to demonstrate fractional statistics of quasiparticles in a macroscopic material. Such abilities, properly extended to large systems, will also be critical for future implementation of naturally fault-tolerant quantum computation.

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- [1] F. Wilczek, Phys. Rev. Lett. **48**, 1144-1146 (1982).
- [2] F. Wilczek, Phys. Rev. Lett. **49**, 957-959 (1982).
- [3] J. Preskill, *Lectures Notes for Physics 219: Quantum Computation* Ch. 9 (Online at <http://www.theory.caltech.edu/people/preskill/ph229/>).
- [4] A. Yu. Kitaev, Ann. Phys. **303**, 2-30 (2003).
- [5] A. Yu. Kitaev, Ann. Phys. **321**, 2-111 (2006).
- [6] J. Preskill, preprint at <http://arxiv.org/abs/quant-ph/9712048> (1997).
- [7] C. Mochon, Phys. Rev. A **67**, 022315 (2003).
- [8] C. Mochon, Phys. Rev. A **69**, 032306 (2004).
- [9] M. H. Freedman, M. Larsen, and Z. -H. Wang, preprint at <http://arxiv.org/abs/quant-ph/0001108> (2000).
- [10] M. H. Freedman, A. Kitaev, and Z. Wang, Comm. Math. Phys. **227**, 587-603 (2002).
- [11] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559-1562 (1982).
- [12] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395-1399 (1983).
- [13] S. D. Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. **94**, 166802 (2005).
- [14] A. Stern, and B. I. Halperin, Phys. Rev. Lett. **96**, 016802 (2006).
- [15] P. Bonderson, A. Kitaev, and K. Shtengel, Phys. Rev. Lett. **96**, 016803 (2006).
- [16] C. Weeks, G. Rosenberg, B. Seradjeh, and M. Franz, Nature Phys. **3** 796-801 (2007).
- [17] L.-M. Duan, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **91**, 090402 (2003).
- [18] Y.-J. Han, R. Raussendorf, and L.-M. Duan, Phys. Rev. Lett. **98**, 150404 (2007).
- [19] Pachos et al., preprint at <http://arxiv.org/abs/0710.0895> (2007).
- [20] C.-Y. Lu et al., preprint at <http://arxiv.org/abs/0710.0278> (2007).
- [21] L. Jiang, G. K. Brennen, A. V. Gorshkov, and K. Hammerer, preprint at <http://arxiv.org/abs/0711>.

- [1365](#)> (2007).
- [22] L. M. K. Vandersypen, and I. L. Chuang, Rev. Mod. Phys. **76**, 1037-1069 (2004).
- [23] E. Fortunato et al., Chem. Phys. **116** (17), 7599 (2002).
- [24] M. A. Pravia et al., J. Chem Phys. **119**, 9993 (2003).
- [25] T. S. Mahesh and D. Suter, Phys. Rev. A **74**, 062312 (2006).